Functional equation in polynomial functions.
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Determine all polynomials $P(x)$ with real coefficients such that
$(x+1)P(x-1)-(x-1)P(x)$ is a constant polynomial.
Solution by Arkady Alt, **San Jose**, **California**, **USA**.
Since $(x+1)P(x-1)-(x-1)P(x)=c$ for any $x\in\mathbb{R}$ and constant $2c$ can be represented in the form $c=\frac{c}{2}(x+1)-\frac{c}{2}(x-1)$ then denoting $b:=\frac{c}{2}$ and $Q(x):=P(x)-b$ we obtain $(x+1)P(x-1)-(x-1)P(x)=c \iff$
$(x+1)\left(P(x-1)-\frac{c}{2}\right)=(x-1)\left(P(x)-\frac{c}{2}\right) \iff (x+1)Q(x-1)=(x-1)Q(x)$
Since $(x-1)Q(x)$ is divisible by $x+1$ and $\gcd(x+1,x-1)=1$ then $Q(x)$ is divisible by $x+1,$that is $Q(x)=R(x)(x+1),$where $R(x)$ is quotient polynomial and, therefore, $(x+1)Q(x-1)=(x-1)Q(x) \iff (x+1)xR(x-1)=(x-1)(x+1)R(x) \iff$
$xR(x-1)=(x-1)R(x).$And again by the same reason as above $S(x):=\frac{R(x)}{x}$ is a polynomial such that $S(x)=S(x-1)$ and $S(x)$ as periodic polynomial with period 1 is a constant polynomial, that is $\frac{R(x)}{x}=a$ for some constant a.
Therefore, $Q(x)=ax(x+1)$ and $P(x)=ax(x+1)+b$.
Checking: $(x+1)P(x-1)-(x-1)P(x)=a(x-1)x(x+1)+b(x+1)-$
$a(x-1)x(x+1)-b(x-1)=2b=c.$
Thus, general solution of given functional equation in polynomial functions is
$P(x)=ax^2+ax+b,$ where $a,b\in\mathbb{R}$.